

More Results on Modular Nim Game

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Abstract

Modular Nim is a two-player impartial combinatorial game played on a directed cycle of n labeled vertices. Two players alternately move a token clockwise to a previously unvisited vertex by a number of steps chosen from a fixed move set $M \subset \mathbb{N}$. Despite being introduced in the 1940s, the game has been solved only in a limited number of cases, and general winning strategies remain largely unknown.

In 2014, Tan and Ward proposed several conjectures about two-move Modular Nim games based on computational evidence. In this work, we investigate these conjectures and provide complete characterizations for several new classes of such games. Furthermore, we give an explicit formulation of a winning strategy for $\Gamma(M; n)$ when the move set M consists solely of odd integers and n is even. Additionally, we identify families of Modular Nim games that are equivalent to previously analyzed cases, thereby extending known results.

Keywords: Combinatorial games; Games on Graphs; Geography; Modular Nim.

1 Introduction

Combinatorial games are two-player games of perfect information with no element of chance, in which players move alternately, and the game ends after a finite number of moves. Modular Nim, also known as Kotzig's Nim, was introduced by Anton Kotzig in 1946 as an impartial combinatorial game. The game can be viewed as a special case of the graph-theoretic game Geography [4]. In Geography, players alternately move a token along directed edges to previously unvisited vertices, and the first player unable to move loses.

Modular Nim can be viewed as a Geography game played on a clockwise directed cycle with n vertices labeled $0, 1, \dots, n-1$ (see Figure 1). Given a move set $M = \{a_1, a_2, \dots, a_s\}$ and a positive integer n , the corresponding Modular Nim game is denoted by $\Gamma(a_1, a_2, \dots, a_s; n)$, or simply by $\Gamma(M; n)$. The rules of the Modular Nim game are as follows:

- A token is initially placed at vertex 0.
- Players take turns moving the token clockwise.
- On each turn, a player moves the token a nodes clockwise, where a is chosen from a predefined set of legal moves $M \subset \mathbb{N}$.
- The token cannot land on any node that has already been visited (or marked).
- The player who is unable to move on his turn loses.

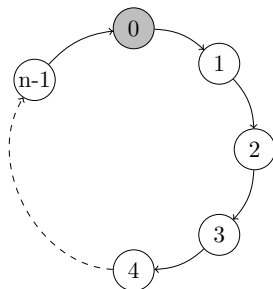


Figure 1: Game setup for Modular Nim

2 Known Results

The first systematic study of Modular Nim appeared in *Winning Ways* [2], which classifies the outcomes of the games $\Gamma(1, 2; n)$ and $\Gamma(1, 3; n)$:

$$P_2 \text{ wins } \Gamma(1, 2; n) \iff n \in \{1, 3, 7\}, \quad P_2 \text{ wins } \Gamma(1, 3; n) \iff n \equiv 1 \text{ or } 3 \pmod{6}.$$

In their foundational work on Modular Nim, Fraenkel et al. [3] established structural reductions that enable the simplification of many instances of the game and also analyzed the more general two-move games with move set of the form $M = \{a, a + 1\}$, using a symmetry-based “diamond” response strategy. They completely solved $\Gamma(2, 3; n)$ and partially solved $\Gamma(3, 4; n)$.

Tan and Ward [5] gave a complete characterization of $\Gamma(1, 4; n)$, and compiled extensive outcome data for all games $\Gamma(a, b; n)$ with $1 \leq a < b \leq 72$ and $n \leq 72$. Based on these computational results, they proposed several conjectures. More recently, Srinivas Arun [1] resolved the conjecture for the game $\Gamma(1, n - 3; n)$ and partially resolved the conjecture concerning the game $\Gamma(1, n - 2; n)$.

3 Our Results

In this work, we resolve several conjectures concerning Modular Nim on directed cycles with two-element move sets, proposed by Tan and Ward in [5]. In particular, we settle the families $\Gamma(a, b; 2(a + b))$, and $\Gamma(2, a; 2 + 2a)$, and establish a general result for $\Gamma(M; k \cdot b)$ when $b \in M$ which yields Conjecture 4.13 from [5] concerning the game $\Gamma(a, b; 2b)$, as a special case.

Furthermore, we complete the classification of the family $\Gamma(2, n - 1; n)$. To achieve this, we introduce the *diamond drop strategy* (see Figure 2), which forms the central idea in the optimal play of the game. Using this approach, we resolve the remaining open cases of a conjecture by Tan and Ward [5], which had been partially settled by Srinivas Arun [1].

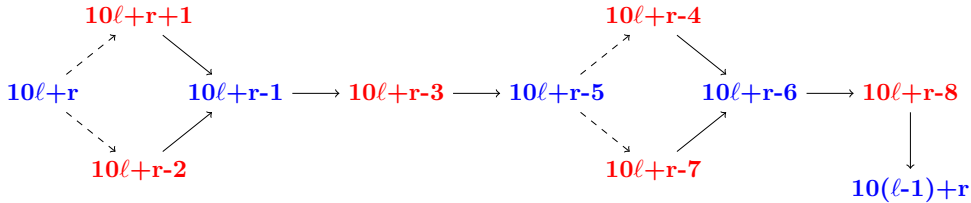


Figure 2: Diamond drop strategy

We also studied new families of games $\Gamma(2, a; 2a - 1)$, $\Gamma(1, a; 2a - 1)$, $\Gamma(1, a; 3a + 1)$, $\Gamma(1, a; 3a - 1)$, and $\Gamma(1, a; 3a - 2)$, inspired by conjectures in [5], and demonstrate their equivalence to previously solved games. Collectively, our results significantly advance the understanding of Modular Nim with two-element move sets.

References

- [1] S. Arun, Geography, Kotzig’s Nim, and variants, *Theor. Comput. Sci.* **1023** (2025).
- [2] E.R. Berlekamp, J.H. Conway, R.K. Guy, *Winning Ways for Your Mathematical Plays*, Vol. 3, *A K Peters, Ltd.*, Natick, MA, 2003.
- [3] A.S. Fraenkel, A. Jaffray, A. Kotzig, G. Sabidussi, Modular Nim, *Theor. Comput. Sci.* **143** (1995) 319–333.
- [4] A.S. Fraenkel, S. Simonson, Geography, *Theor. Comput. Sci.* **110** (1993) 197–214.
- [5] X.L. Tan, M.D. Ward, On Kotzig’s Nim, *Integers* **14** (2014) #G6.